Please note that 10 minutes reading time is included in the 3 hours allowed for this examination.

You should attempt all nine questions in Part I. All of these questions carry the same number of marks and Part I comprises 63% of the total marks available.

You should answer TWO questions from Part II and ONE question from Part III. Each question in Part II carries 11 marks and each question in Part III carries 15 marks. Record all your answers in the answer book(s) provided.

At the end of the examination
Remember to write your name, personal identifier and examination number on each answer book used. Failure to do so will mean that your papers cannot be identified. Attach your answer books together using the fastener provided.
PART I

You should attempt ALL questions in this part of the examination.

You may use the following facts about series:

\[ \sum_{r=0}^{n} x^r = \frac{1 - x^{n+1}}{1 - x}, \quad \text{and so} \quad \sum_{r=0}^{\infty} x^r = \frac{1}{1 - x}, \quad |x| < 1. \]

You may need the following indefinite integrals in this and later parts of the examination:

\[ \int \frac{1}{(a + bx)^{1/2}} \, dx = \frac{2}{b} (a + bx)^{1/2}; \]

\[ \int e^{ax} \cos bx \, dx = \frac{1}{a^2 + b^2} (a \cos bx + b \sin bx) e^{ax}. \]

Question 1 – 7 marks

(a) Show directly from the definition of a null set that the set

\[ S = \left\{ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots \right\} \]

is null.

(b) Is the set

\[ T = [0, 1] \setminus S \]

null? Briefly justify your answer, quoting any result that is relevant. No calculations are necessary.

Question 2 – 7 marks

Let

\[ \varphi = -2 \chi_{(-1, 1/2]} + 3 \chi_{[0, 2)} - \frac{1}{2} \chi_{(1/2, 1)}. \]

(a) Sketch the graph of \( \varphi \).

(b) Express \( \varphi \) as a linear combination of characteristic functions of disjoint intervals.

(c) Evaluate \( \int \varphi \) by using the expression obtained in part (b). Would it make any difference to \( \int \varphi \) had you used the original expression? Comment on your answer.

Question 3 – 7 marks

For each integer \( n = 0, 1, 2, \ldots \), let

\[ \varphi_n = \frac{1}{2} \sum_{r=0}^{n} \frac{1}{9^r} \chi_{[3r, 3r+1[}. \]

(a) Sketch the graphs of \( \varphi_0, \varphi_1 \) and \( \varphi_2 \). Show that the sequence of functions \( \{\varphi_n\} \) is increasing.

(b) Evaluate \( \int \varphi_n \).

(c) Show that the sequence \( \{\int \varphi_n\} \) is bounded above.

(d) Use the definition of \( L^\infty(\mathbb{R}) \) to show that the sequence of functions \( \{\varphi_n\} \) converges a.e. to a function \( f \in L^\infty(\mathbb{R}) \), and hence determine \( \int f \). (It is not necessary to determine \( f \) explicitly in order to do the question, and no credit will be given for doing so.)
Question 4 – 7 marks

Consider the sequence of step functions \( \{ \varphi_n : n \geq 1 \} \) defined by

\[ \varphi_n = \frac{1}{n^2} \chi_{[1,n^2]} . \]

(a) Sketch the graphs of \( \varphi_1, \varphi_2, \varphi_3 \) and \( \varphi_4 \).

(b) Verify that the sequence of functions \( \{ \varphi_n \} \) converges to the zero function

\[ f(x) = 0, \quad x \in \mathbb{R}, \]

pointwise everywhere, but that

\[ \int f \neq \lim_{n \to \infty} \int \varphi_n . \]

(c) Why would it be false to say that the inequality in part (b) is a contradiction because, by the result on \( L^\infty(\mathbb{R}) \) (Handbook page 15),

\[ \int f = \lim_{n \to \infty} \int \varphi_n ? \]

Question 5 – 7 marks

Consider the sequence of step functions \( \{ \varphi_n : n \geq 1 \} \) with \( \varphi_n : \mathbb{R}^2 \to \mathbb{R} \) defined by

\[ \varphi_n(x,y) = \sum_{r=1}^{n} \chi_{[0,2^{-n}]}(x) \chi_{[0,2^{-n}]}(y). \]

(a) Sketch the regions in the \((x,y)\)-plane where \( \varphi_n(x,y) \neq 0 \) for \( n = 1, 2 \) and \( 3 \).

(b) Show that \( \{ \varphi_n \} \) is increasing.

(c) Determine \( \int \varphi_n \). Does the sequence \( \{ \int \varphi_n \} \) converge, and if so, what is its limit?

(d) Briefly justify the statement that \( \{ \varphi_n(x,y) \} \) converges a.e. to a function \( f \in L^1(\mathbb{R}^2) \) and determine \( \int f \).

Question 6 – 7 marks

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[ f(x,y) = \begin{cases} y, & \text{if } 0 < x < 1; \\ -y, & \text{if } -1 < x < 0; \\ 0, & \text{otherwise}. \end{cases} \]

Show that

\[ \int \left( \int f(x,y) \, dx \right) \, dy = 0, \]

but that

\[ \int \left( \int |f(x,y)| \, dx \right) \, dy \]

does not exist.

What does Fubini’s Theorem then tell you about the integrability of \( f \)?
Question 7 – 7 marks
Let \( f \) be the function defined by
\[
f(x) = (1 - x)^{-1/2} \chi_{[0,1]}, \quad x \in \mathbb{R}.
\]
By considering the sequence \( \{ f \chi_{[0,1/1]} \} \) of restrictions of \( f \), use the Monotone Convergence Theorem to show that \( f \in L^1(\mathbb{R}) \). You may find it useful to consider W Theorem 3.3.1 (Handbook page 16). Why does this theorem not enable us to deduce at once that \( f \in L^1(\mathbb{R}) \)?

Question 8 – 7 marks
Consider the function
\[
f(x) = x^{-1/2} \chi_{(0,\infty)}.
\]
You may assume that \( f \) is not integrable, but that \( f \chi_{(a,b]} \in L^1((a,b]) \) for all \( 0 \leq a < b < \infty \).

(a) By finding an explicit form for the set
\[
A_c = \{ x \in \mathbb{R} : f(x) \geq c \}, \quad c \in \mathbb{R},
\]
show that \( A_c \) is measurable and find its measure. What does this tell you about the measurability of the function \( f \)?

(b) By truncating \( f \) at both ends, or otherwise, demonstrate that it can be written as the limit pointwise everywhere of a sequence of integrable functions.

(c) What condition is necessary for the application of W Proposition 5.1.1 (Handbook page 22) that this example does not possess?

Question 9 – 7 marks
Let \( E_1, E_2 \) and \( E_3 \) be measurable subsets of \( \mathbb{R} \) of finite measure. As usual, we write
\[
m(E) = \int \chi_E
\]
for the measure of a measurable set \( E \).

(a) Let
\[
\sigma_1 = m(E_1) + m(E_2), \\
\sigma_2 = m(E_1 \cap E_2).
\]
Using only the identity
\[
\chi_S \cup T + \chi_S \cap T = \chi_S + \chi_T
\]
for measurable sets \( S \) and \( T \), show that
\[
m(E_1 \cup E_2) = \sigma_1 - \sigma_2.
\]

(b) Now define
\[
\tau_1 = m(E_1) + m(E_2) + m(E_3), \\
\tau_2 = m(E_1 \cap E_2) + m(E_1 \cap E_3) + m(E_2 \cap E_3), \\
\tau_3 = m(E_1 \cap E_2 \cap E_3).
\]
Using only the above identity and the set identities
\[
R \cap (S \cup T) = (R \cap S) \cup (R \cap T), \\
(R \cap S) \cap (R \cap T) = R \cap S \cap T,
\]
express \( m(E_1 \cup E_2 \cup E_3) \) in terms of \( \tau_1, \tau_2 \) and \( \tau_3 \). (If you set \( E_3 = \emptyset \) in your answer, it must coincide with the expression in part (a).)
PART II

You should attempt TWO questions in this part of the examination.

Question 10 — 11 marks

The function \( f_n : [0,1] \rightarrow \mathbb{R} \) is defined by
\[
f_n(x) = nx e^{-nx^2}, \quad n = 1, 2, 3, \ldots
\]

(a) Use the Taylor series for \( e^t \) around \( t = 0 \) to show that the sequence \( \{f_n\} \) converges pointwise, and find the function \( f : [0,1] \rightarrow \mathbb{R} \) to which it converges.

(b) Find the maximum value of the function \( F : [0, \infty) \rightarrow \mathbb{R} \) defined by
\[
F(t) = te^{-t^2/2}.
\]
Using this result, find a bound for the \( \{f_n\} \) which is independent of \( n \) and \( x \in [0,1] \). Then use the Bounded Convergence Theorem to find
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx.
\]

Question 11 — 11 marks

Let \( \{E_n : n \geq 1\} \) be a countably infinite sequence of measurable subsets of \( \mathbb{R} \) with the property that
\[
\sum_{n=1}^{\infty} m(E_n) = M < \infty.
\]
Here
\[
m(E) = \int \chi_E
\]
is the measure of the measurable set \( E \).

Prove that the set \( A \) of points \( x \in \mathbb{R} \) which belong to an infinity of the sets \( E_n \) has measure zero. Put differently, if \( x \in \mathbb{R} \) belongs to at most a finite number of the sets \( E_n \), then \( x \) belongs to the complement of \( A \), and we want you to prove that \( A \) is measurable, with \( m(A) = 0 \).

We suggest that you consider the sequence of functions \( \{\varphi_n\} \) defined by
\[
\varphi_n = \sum_{k=1}^{n} \chi_{E_k},
\]
and that you pay particular attention to the points at which this sequence does not converge pointwise.
Question 12 – 11 marks

Let \( I \) be a closed and bounded subset of \( \mathbb{R} \). Let \( \{f_n\} \) be a sequence of continuous functions, and suppose this sequence converges uniformly on \( I \) to a function \( f \). Prove that for \( p \geq 1 \) the sequence \( \{f_n\} \) converges to \( f \) in the \( L^p(I) \)-norm and \( f \in L^p(I) \) by the following method.

You are reminded that a sequence \( \{f_n\} \) of functions on an interval \( I, f_n : I \to \mathbb{R} \), converges uniformly to the function \( f : I \to \mathbb{R} \) if

\[
\lim_{n \to \infty} \sup_{x \in I} |f(x) - f_n(x)| = 0,
\]

and that if a sequence of continuous functions converges uniformly, the limit function is continuous. You are also reminded that

\[
\|f\| = \sup_{x \in I} |f(x)|
\]

is a norm (the uniform norm).

(a) First show that \( f_n \in L^p(I) \). Then show that \( f \in L^p(I) \).

(b) Writing the uniform convergence of \( \{f_n\} \) to \( f \) in terms of the uniform norm, combine this with the properties of the integral to show that \( \|f - f_n\|_p \) converges to 0.

Question 13 – 11 marks

Let \( f : \mathbb{R} \to \mathbb{R} \) be the function of period \( 2\pi \) that coincides with

\[ e^{ix}/\sqrt{2} \chi_{(-\pi, \pi]}(x). \]

on the interval \( (-\pi, \pi] \).

(a) Justify the statement: \( f \in L^2([\pm \pi, \pi]) \).

(b) Determine the Fourier series of \( f \).

(c) By reference to the appropriate items in the Handbook, explain briefly how you know that the Fourier series for \( f \) converges in the mean and pointwise. Omit the points \( \pm \pi \) from consideration when examining pointwise convergence.

(d) Using the Fourier series for \( f \), show that

\[
\sqrt{2} U V = \frac{4U^2}{\pi} + \frac{2U^2}{\pi} \sum_{k=2}^{\infty} \left( \frac{1}{k^2 + \frac{1}{2}} \right)^2 + \frac{2V^2}{\pi} \sum_{k=1}^{\infty} \left( \frac{1}{k^2 + \frac{1}{2}} \right)^2.
\]

We have introduced the notation

\[
U = \left( e^{ix}/\sqrt{2} - 1 \right) \quad \text{and} \quad V = \left( e^{ix}/\sqrt{2} + 1 \right),
\]

which we should like you to use in your solution. Results on pointwise convergence of Fourier series may be assumed, but should be clearly stated. Parseval’s equation may be assumed.
PART III

You should attempt ONE question in this part of the examination.

Question 14 – 15 marks

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) be functions which are continuous and integrable. Show that

\[
h(s) = \int f(s - t)g(t) \, dt
\]

defines an integrable function \( h : \mathbb{R} \rightarrow \mathbb{R} \) almost everywhere, and that

\[
\|h\| \leq \|f\| \|g\|
\]

where for any integrable function \( k \),

\[
\|k\| = \int |k(r)| \, dr
\]

is the \( L^1 \) norm of \( k \).

We suggest that you begin by defining functions \( \alpha : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( \beta : \mathbb{R}^2 \rightarrow \mathbb{R} \) such that

\[
f \circ \alpha(s,t) = f(s - t) \quad \text{and} \quad g \circ \beta(s,t) = g(t),
\]

and then proving that \( f \circ \alpha \) and \( g \circ \beta \) are continuous. You will then be able to show that an appropriate theorem on integration over \( \mathbb{R}^2 \) applies to the function

\[
H(s,t) = f(s-t)g(t),
\]

from which you will be able to conclude that \( H \) is integrable. Using the theorem again, you will be able to estimate \( \|h\| \).

The following result, which you may assume without proof, is crucial in the proof. If \( k : \mathbb{R} \rightarrow \mathbb{R} \) is integrable, then for any \( a \in \mathbb{R} \), so is the function \( k_a : \mathbb{R} \rightarrow \mathbb{R} \) defined for almost all \( r \in \mathbb{R} \) by

\[
k_a(r) = k(r - a).
\]

Moreover,

\[
\int k_a = \int k.
\]

This result is known as the translational invariance of Lebesgue measure, and \( h \) is known as the convolution of \( f \) and \( g \).
Question 15 - 15 marks

Let \( \{f_n\} \) be a sequence of functions in \( L^2(\mathbb{R}) \) which has the property that the series of positive terms

\[
\sum_{n=1}^\infty \|f_n\|
\]

converges, and let

\[
A = \sum_{n=1}^\infty \|f_n\|.
\]

Using the method that follows, prove that the series of functions \( \sum_{n=1}^\infty f_n \) converges absolutely almost everywhere, and that if \( f \) is a function such that

\[
f = \sum_{n=1}^\infty f_n \text{ a.e.},
\]

then \( f \in L^2(\mathbb{R}) \) and

\[
\|f\| = \lim_{n \to \infty} \left\| \sum_{k=1}^n f_k \right\| \leq A.
\]

In this problem, for any function \( g \in L^2(\mathbb{R}) \), \( \|g\| \) denotes the \( L^2 \)-norm of \( g \).

(a) First show that

\[
\int \left( \sum_{k=1}^n |f_k| \right)^2 \leq \left( \sum_{k=1}^n \|f_k\| \right)^2 \leq A^2.
\]

(b) Apply the Monotone Convergence Theorem to prove that

\[
\int h \leq A^2,
\]

where \( h \) is defined by

\[
h(x) = \begin{cases} 
\lim_{n \to \infty} (\sum_{k=1}^n |f_k(x)|)^2, & \text{when the limit exists,} \\
0, & \text{otherwise.}
\end{cases}
\]

(c) Now use the Dominated Convergence Theorem to show that \( f^2 \) is integrable. Combining the various results and inequalities, complete the demonstration.

[END OF QUESTION PAPER]
M431 Solutions to Specimen Examination Paper

Part I

The student should have attempted ALL questions in this part of the examination.
The scales on the diagrams are not standard, having been chosen for legibility in each case.

Question 1

(a) Let $\varepsilon > 0$ be given. Surround the point $2^{-n}$ by the open interval

$$I_n = \left(\frac{1}{2^n} - \frac{\varepsilon}{3^n}, \frac{1}{2^n} + \frac{\varepsilon}{3^n}\right).$$

The family

$$\{I_n : n = 1, 2, 3, \ldots\}$$

covers the set $S$, as each point of $S$ is the centre of the interval $I_n$:

$$S \subseteq \bigcup_{n=1}^{\infty} I_n.$$  

The length of $I_n$ is

$$l(I_n) = 2 \cdot \frac{\varepsilon}{2^n} = \frac{\varepsilon}{3^n}.$$  

The total length of the sequence is (Weir, page 18)

$$\sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{3^n} = \varepsilon \left(\frac{1}{1-\frac{1}{3}} - 1\right) = \varepsilon/2 < \varepsilon.$$  

As $\varepsilon$ was arbitrary, this proves that $S$ is null.

(Note that we were asked to use the definition directly, as we have done. It would not have been enough here to quote the result that all countable sets are null.)

(b) Were $T$ null, then $S \cup T = [0,1]$ would be null (W Proposition 2.2.3, Handbook, page 4). Were $[0,1]$ null, its measure, which is (item 1 under 'Measurable sets and measure', Handbook, page 25)

$$m([0,1]) = \int \chi_{[0,1]} = 1,$$

would be 0 (W Theorem 6.2.1, (a) (i), Handbook, page 26), which it is not. Hence $T$ is not null.
Question 2

(a)

(b) From the diagram,
\[ \varphi = -2x_{(-1,0)} + x_{[0,1/2]} + \frac{5}{2}x_{(1/2,1)} + 3x_{(1,2)} \]

(c)
\[ \int \varphi = -2[0 - (-1)] + \left( \frac{1}{2} - 0 \right) + \frac{5}{2} \left( 1 - \frac{1}{2} \right) + 3(2 - 1) = \frac{11}{4}. \]

It would make no difference to \( \int \varphi \) had we used the original expression, because the integral of a step function does not depend on its expression as a linear combination of characteristic functions.

Question 3

(a)
The sequence is increasing, because

\[ \varphi_{n+1} = \varphi_n + \frac{1}{2} \left( \frac{1}{9^{n+1}} + \frac{1}{3^{n+1}} \right) \geq \varphi_n. \]

(b) \[
\int \varphi_n = \frac{1}{2} \sum_{r=0}^{n} \frac{1}{9^r} (3^{r+1} - 3^r)
= \sum_{r=0}^{n} \frac{1}{9^r} (3^r)
= \sum_{r=0}^{n} \frac{1}{3^r}
= \frac{3}{2} \left[ 1 - \left( \frac{1}{3} \right)^{n+1} \right],
\]
using the result given at the beginning of the paper.

(c) This sequence is bounded above, with

\[ \frac{3}{2} \left[ 1 - \left( \frac{1}{3} \right)^{n+1} \right] < \frac{3}{2}. \]

Hence \( \int \varphi_n < \frac{3}{2}. \)

(d) As \( \{ \varphi_n \} \) is an increasing sequence of step functions for which \( \{ \int \varphi_n \} \) is bounded above, \( \{ \varphi_n \} \) converges a.e. to a function \( f \in L^{\infty}(\mathbb{R}) \), with

\[ \int f = \lim_{n \to \infty} \int \varphi_n = \frac{3}{2}. \]

**Question 4**

(a)
(b) For \( x < 1 \), \( \varphi_n(x) = 0 \) for all \( n \), so consider \( x \geq 1 \). For any \( x > 1 \), choose \( n \) such that \( n^2 < x \leq (n+1)^2 \). Then \( \varphi_k(x) = 0 \) for \( 1 \leq k \leq n \), and \( \varphi_k(x) = 1/k^2 \) for \( k \geq n + 1 \). Then

\[
\lim_{k \to \infty} \varphi_k(x) = 0, \quad x > 1.
\]

For \( x = 1 \), we have \( \varphi_n(1) = n^{-2} \), so

\[
\lim_{k \to \infty} \varphi_k(1) = 0,
\]

and so the sequence \( \{\varphi_n\} \) converges to the zero function \( f \) pointwise everywhere.

Now \( \int f = 0 \), and

\[
\int \varphi_n = \frac{n^2 - 1}{n^2} = 1 - \frac{1}{n^2}.
\]

Therefore

\[
\lim_{n \to \infty} \int \varphi_n = 1 \neq \int f = 0.
\]

(c) It is true that, as \( f \in L^{\infty}(\mathbb{R}) \), it can be written as the limit of an increasing sequence of step functions \( \{\psi_n\} \) converging to \( f \) a.e., and for which the sequence \( \{\int \psi_n\} \) converges to \( \int f \). However, the sequence \( \{\varphi_n = n^{-2} \chi_{[1,n^2]}\} \) is not increasing. For example,

\[
\varphi_1(3) = 0, \quad \varphi_2(3) = \frac{1}{4}, \quad \varphi_3(3) = \frac{1}{6}.
\]

**Question 5**

(a)
(b) As \( \varphi_{n+1}(x,y) = \varphi_n(x,y) + \chi_{[0,1/4^{n+1}]}(x)\chi_{[0,2^{n+1}]}(y) \geq \varphi_n(x,y) \), the sequence \( \{ \varphi_n \} \) is increasing.

c) The integrals are
\[
\int \varphi_n = \sum_{r=1}^{n} (2^{-2r}) (2^r) = \sum_{r=1}^{n} 2^{-r}.
\]
From the series given at the beginning of the paper,
\[
\int \varphi_n = 1 - \left( \frac{1}{2} \right)^n,
\]
and so
\[
\lim_{n \to \infty} \int \varphi_n = 1.
\]
Thus the sequence \( \{ \int \varphi_n \} \) converges, and has the limit 1.

d) The sequence \( \{ \varphi_n \} \) is an increasing sequence of step functions on \( \mathbb{R}^2 \) for which the sequence \( \{ \int \varphi_n \} \) converges. Therefore (W Theorem 4.2.1, Handbook, page 18) \( \{ \varphi_n \} \) converges almost everywhere to a function \( f \), and (W page 78, Handbook, page 19) \( f \in L^1(\mathbb{R}^2) \), with
\[
\int f = \lim_{n \to \infty} \int \varphi_n = 1.
\]

Question 6

For each \( y \in \mathbb{R} \),
\[
\int f(x,y) \, dx = -\int_{-1}^{y} y \, dx + \int_{0}^{1} y \, dx = -y + y = 0.
\]
Hence
\[
\int \left( \int f(x,y) \, dx \right) dy = 0.
\]
But for each \( y \in \mathbb{R} \),
\[
\int |f(x,y)| \, dx = \int_{-1}^{y} y \, dx + \int_{0}^{1} y \, dx = y + y = 2y.
\]
The function \( F(y) = 2y \) is not in \( L^\infty(\mathbb{R}) \), so
\[
I = \int \left( \int |f(x,y)| \, dx \right) dy
\]
does not exist.

Were \( f \) integrable, \( |f| \) would be integrable and \( I \) would exist, by Fubini's Theorem. Hence \( f \notin L^1(\mathbb{R}) \).

Question 7

We are told to consider
\[
f_n(x) = (1-x)^{-1/2} \chi_{[0,1-1/n]}(x), \quad x \in \mathbb{R}.
\]
Now \( f_n \) is continuous on \([0,1-1/n]\) and vanishes outside \([0,1-1/n]\), so by W Theorem 3.3.1 (Handbook page 16) it is integrable. Moreover
\[
\int f_n = \int_{0}^{1-1/n} (1-x)^{-1/2} \, dx.
\]
Using the indefinite integral given at the beginning of this paper we have, by the Fundamental Theorem of Calculus,
\[
\int f_n = \left[ -2(1-x)^{1/2} \right]_{0}^{1-1/n} = 2 \left[ 1 - \left( \frac{1}{n} \right)^{1/2} \right] < 2.
\]
Since the sequence \( \{ f_n \} \) of integrable functions is increasing and \( \{ \int f_n \} \) is bounded, it follows from the Monotone Convergence Theorem that \( \{ f_n \} \) converges a.e. to an integrable function. But \( \{ f_n \} \) converges pointwise to \( f \), so \( f \in L^1(\mathbb{R}) \).
Question 8

(a) The sets $A_c$ are

$$A_c = \begin{cases} \mathbb{R}, & \text{if } c \leq 0, \\ \left[0, \frac{1}{c^2}\right], & \text{if } c > 0. \end{cases}$$

These are measurable sets, with

$$m(A_c) = \begin{cases} \infty, & \text{if } c \leq 0, \\ \frac{1}{c^2}, & \text{if } c > 0. \end{cases}$$

By Proposition 6.2.1 (Handbook, page 26) $f$ is measurable.

(b) We consider $f_n = f \chi_{I_n}$, where $I_n = (1/n, n)$, or some variant of this. We see that $f_n \in L^1(I_n)$, since $I_n$ is bounded. Moreover, the sequence $\{f_n\}$ converges pointwise to $f$.

(c) We can write

$$f' = f.$$ 

The quoted proposition does not apply to this example, as the integrals

$$\int_{I_n} |f| = \int_{1/n}^{n} x^{-1/2} dx = 2n^{1/2} - \frac{2}{n^{1/2}}$$

are not bounded above.

Question 9

(a) This is immediate from taking the integral of the identity, after setting $S = E_1$ and $T = E_2$:

$$\chi_{E_1} \cup E_2 = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

implies

$$\int \chi_{E_1} \cup E_2 = \int \chi_{E_1} + \int \chi_{E_2} - \int \chi_{E_1 \cap E_2},$$

or

$$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2) = \sigma_1 - \sigma_2.$$ 

(b) By setting $S = E_1$ we get the intermediate result

$$\chi_{E_1} \cup T = \chi_{E_1} + \chi_T - \chi_{E_1 \cap T}.$$ 

Now set $T = E_2 \cup E_3$. Then

$$\chi_T = \chi_{E_2} + \chi_{E_3} - \chi_{E_2 \cap E_3}.$$ 

The first set identity given yields

$$\chi_{E_1 \cap T} = \chi_{E_1} \cap (E_2 \cup E_3) = \chi_{E_1} \cap (E_2 \cup E_3) = \chi_{E_1} \cap E_2 + \chi_{E_1} \cap E_3 - \chi_{(E_1 \cap E_2) \cap (E_1 \cap E_3)}.$$ 

The second set identity now yields

$$\chi_{(E_1 \cap E_2) \cap (E_1 \cap E_3)} = \chi_{E_1} \cap E_2 \cap E_3.$$ 

Being careful with minus signs, it is now immediate that

$$m(E_1 \cup E_2 \cup E_3) = \tau_1 - \tau_2 + \tau_3.$$ 

As noted in the question, suppose $E_3 = \emptyset$. Then $\tau_3 = 0, \tau_1 = \sigma_1$ and $\tau_2 = \sigma_2$, and so this result is consistent with the answer in part (a).
Part II
The student should have attempted TWO questions in this part of the examination

Question 10

(a) The Taylor series for $e^t$ is

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

and is certainly greater than any of its terms if $t > 0$. In particular,

$$e^t > \frac{t^3}{3!}.$$  

For $x \in (0, 1]$ set $t = (nx)^{1/2}$, and so

$$f_n(x) = \frac{nx}{e^t} < \frac{nx}{t^3/3!} = \frac{3!nx}{(nx)^{3/2}}.$$  

Thus

$$\lim_{n \to \infty} f_n(x) = 0, \quad x \neq 0.$$  

As $f_n(0) = 0$ for all $n$, we conclude that the sequence $\{f_n\}$ converges pointwise to the zero function.

(b) For $t > 0$,

$$F'(t) = e^{-t/2} \left[1 - \frac{t}{2}t^{1/2}\right],$$

so

$$F'(t) \begin{cases} > 0, & \text{if } t < 4, \\ = 0, & \text{if } t = 4, \\ < 0, & \text{if } t > 4. \end{cases}$$

Then $F$ has a maximum at $t = 4$, at which its value is

$$F(4) = 4e^{-2}.$$  

Then for any $x \in [0, 1]$,

$$0 \leq f_n(x) = F(nx) \leq 4e^{-2}.$$  

This bound is independent of both $n$ and $x$.

Each $f_n \in L^1([0, 1])$ by W Theorem 4.2.4 (Handbook, page 20), the sequence $\{f_n\}$ converges pointwise almost everywhere to the zero function, and there is a positive constant $K = 4e^{-2}$ such that

$$|f_n(x)| \leq K, \quad x \in [0, 1], \quad n \in \mathbb{N}.$$  

Hence by the Bounded Convergence Theorem, W page 110 (Handbook, page 24)

$$\lim_{n \to \infty} \int_0^1 f_n = \int_0^1 \lim_{n \to \infty} f_n = 0.$$  

Question 11

Because each $E_k$ is measurable and

$$m(E_k) \leq \sum_{n=1}^{\infty} m(E_n) = M < \infty,$$

each function $\chi_{E_k}$ is integrable. Defining $\varphi_n$ as in the question, it is a sum of $n$ integrable functions, and so is integrable. It is a positive simple function, and

$$\varphi_{n+1} = \varphi_n + \chi_{E_{n+1}} \geq \varphi_n,$$

so the sequence of simple integrable functions $\{\varphi_n\}$ is increasing.

We observe that

$$0 \leq \int \varphi_n \leq \sum_{k=1}^{n} m(E_k) \leq M < \infty,$$

so the sequence $\{\int \varphi_n\}$ is bounded. By the Monotone Convergence Theorem,
Theorem 5.1.2 (Handbook, page 22), the sequence \( \{\varphi_n\} \) converges almost everywhere to a function \( f \in L^1(\mathbb{R}) \), with

\[
\int f = \lim_{n \to \infty} \int \varphi_n \leq \sum_{k=1}^{\infty} m(E_k) = M.
\]

Consider a point \( x \in \mathbb{R} \) for which the sequence \( \{\varphi_n\} \) converges. If \( x \) belongs to \( m \) sets \( E_k \), then \( f(x) = m \), and vice versa.

If \( x \in \mathbb{R} \) is a point for which the sequence \( \{\varphi_n\} \) does not converge, this can only be because the number of terms contributing to the sum is not finite. Hence \( x \) must belong to the set \( A \). Conversely, if \( x \in A \), it must contribute countably infinitely many 1s to the sum and so be a point at which the sequence does not converge.

Thus \( A \) is exactly the set of points at which convergence of the sequence \( \{\varphi_n\} \) does not occur. The Monotone Convergence Theorem assures us that the convergence occurs except for a set of measure zero, so \( A \) has measure zero. (This is why we used the Monotone Convergence Theorem in what appears to be a problem purely about sets.)

Question 12

(a) If \( h \) is a continuous function on \( I \), then so is the function obtained by taking its \( p \)th power: \( x \to h(x)^p \) is continuous. Applying this to \( f_n \) and using W page 48 (Handbook, page 16), we conclude that \( f_n \in L^p(I) \). As \( \{f_n\} \) converges uniformly to \( f \), \( f \) is continuous on \( I \); we use W page 48 (Handbook, page 16) to conclude that \( f \in L^p(I) \).

(b) We know that \( L^p(I) \) is a vector space, so \( f - f_n \in L^p(I) \), and so the integral \( \|f - f_n\|_p \) exists.

Writing

\[
\|h\| = \sup\{|h(x)| : x \in I\}
\]

for the uniform norm, we observe that

\[
|h(x)| \leq \|h\| \quad \text{for all } x \in I.
\]

Another property of the supremum that we use is

\[
|h(x)|^p \leq \|h\|^p \leq \|h\|^p \quad \text{for all } x \in I.
\]

Now we use the fact that if \( h \leq k \) on \( I \), where \( k \) is integrable on \( I \), then

\[
\int h \leq \int k.
\]

Take \( h \) to be \( f - f_n \) and \( k \) to be the constant function \( ||f - f_n||_p \). Then the \( p \)th power of \( ||f - f_n||_p \) can be estimated as

\[
(||f - f_n||_p)^p = \int |f(x) - f_n(x)|^p
\]

\[
\leq \int ||f - f_n||_p^p
\]

\[
= m(I)||f - f_n||_p^p,
\]

where \( m(I) \) is the measure of the set \( I \).

Now taking the \( p \)th root and then the limit as \( n \to \infty \),

\[
\lim_{n \to \infty} ||f - f_n||_p \leq m(I)^{1/p} \lim_{n \to \infty} ||f - f_n|| = 0.
\]

This completes the solution.
We are given the function
\[ f(x) = e^{\frac{|x|}{\sqrt{2}}} \chi_{(-\pi, \pi]}(x), \]
and periodically extended to all of \( \mathbb{R} \).

(a) Consider the function
\[ g = f(x)^2 \chi_{(-\pi, \pi]}. \]
This function vanishes outside the interval \([-\pi, \pi]\), is bounded on that interval, and is continuous there. By W Theorem 3.3.1 (Handbook, page 16), \( g \) is integrable. The function \( f \) is continuous on \([-\pi, \pi]\) and so measurable there, by W Theorem 6.1.1 (Handbook, page 25). Hence (W pages 164–6, Handbook, page 27) \( f \in L^2([-\pi, \pi]) \).

(b) All the sine integrals vanish because \( f \) is even.
\[ \frac{1}{2} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\frac{|x|}{\sqrt{2}}} \, dx = \frac{1}{\pi} \sqrt{2} U. \]

For \( k \geq 1 \) we use the cosine integral given at the beginning of this paper to obtain
\[ a_k = \frac{2}{\pi} \int_{0}^{\pi} e^{\frac{|x|}{\sqrt{2}}} \cos kx \, dx \]
\[ = \frac{2}{\pi} \frac{1}{k^2 + \frac{1}{2}} \left[ e^{\pi/\sqrt{2}} \left( k \sin(kx) + \frac{1}{\sqrt{2}} \cos(kx) \right) \right]_{0}^{\pi} \]
\[ = \frac{\sqrt{2}}{\pi} \frac{1}{k^2 + \frac{1}{2}} \left[ (-1)^k e^{\pi/\sqrt{2}} - 1 \right]. \]

Using the symbols \( U \) and \( V \),
\[ f(x) \sim \frac{\sqrt{2} U}{\pi} + \frac{\sqrt{2} U}{\pi} \sum_{k=2}^{\infty} \frac{1}{k^2 + \frac{1}{2}} \cos(kx) - \frac{\sqrt{2} V}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2 + \frac{1}{2}} \cos(kx) \]
is the Fourier series of \( f \) (W page 204, Handbook, page 33).

(c) Because \( f \in L^2([-\pi, \pi]) \), the series converges in the mean.

As \( f \) is increasing on \([0, \pi]\) and decreasing on \((-\pi, 0]\), it is of bounded variation there. By the same argument given in part (a), \( f \in L^1([-\pi, \pi]) \). Hence we can apply Jordan’s Theorem to conclude that its Fourier series converges at these points.

(We remark, parenthetically, that W Proposition 7.6.3 does not apply here as \( e^{\frac{|x|}{\sqrt{2}}} \) is not a step function. The points \( \pm \pi \) are omitted because you have no way of knowing that \( f \) is of bounded variation on intervals \([\pm \pi - r, \pm \pi + r]\).)

(d) Because of mean convergence, proved in part (c), we can use Parseval’s equation (Theorem T.12.2.2, Handbook, page 32) for the series. First we calculate
\[ f \chi_{(-\pi, \pi]} \cdot f \chi_{(-\pi, \pi]} = 2 \int_{0}^{\pi} e^{\sqrt{2} \pi} \, dx \]
\[ = \sqrt{2} \left( e^{\sqrt{2} \pi} - 1 \right) \]
\[ = \sqrt{2} UV. \]

Using the orthonormality of the functions (Handbook, page 34)
\[ e_1(x) = \frac{1}{\sqrt{2\pi}}, \quad e_{2k+1}(x) = \frac{1}{\sqrt{\pi}} \cos kx \quad (k = 1, 2, \ldots), \]
we find that
\[ \sqrt{2} UV = \frac{4U^2}{\pi} + \frac{2U^2}{\pi} \sum_{k=2}^{\infty} \left( \frac{1}{k^2 + \frac{1}{2}} \right)^2 + \frac{2V^2}{\pi} \sum_{k=1}^{\infty} \left( \frac{1}{k^2 + \frac{1}{2}} \right)^2. \]

This is the equation we were required to derive.
Question 14

Define \( \alpha : \mathbb{R}^2 \rightarrow \mathbb{R} \) by \( \alpha(s, t) = s - t \) and \( \beta : \mathbb{R}^2 \rightarrow \mathbb{R} \) by \( \beta(s, t) = t \). Then
\[
f \circ \alpha(s, t) = f(s - t) \quad \text{and} \quad g \circ \beta(s, t) = g(t).
\]

Now \( \alpha \) and \( \beta \) will be shown to be continuous. First we consider \( \alpha \):
\[
|\alpha(s, t) - \alpha(s', t')| = |(s - s') - (t - t')|
\leq |s - s'| + |t - t'|
= \sqrt{(s - s')^2 + (t - t')^2}
\leq 2\sqrt{(s - s')^2 + (t - t')^2}.
\]

As this last expression is twice the Euclidean distance between the points \((s, t)\) and \((s', t')\) in \(\mathbb{R}^2\), continuity of \(\alpha\) is assured.

Similarly, for \( \beta \) we find that
\[
|\beta(s, t) - \beta(s', t')| = |(t - t')|
= \sqrt{(t - t')^2}
\leq \sqrt{(s - s')^2 + (t - t')^2},
\]
proving the continuity of \( \beta \).

We now show that if \( \psi : \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous and \( k : \mathbb{R} \rightarrow \mathbb{R} \) is continuous, then \( k \circ \psi : \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous. In this case it is easier to use the sequence definition of continuity. Let \((s, t)\) be any point in \(\mathbb{R}^2\) and \((s_n, t_n)\) any sequence in \(\mathbb{R}^2\) converging to \((s, t)\). Then \(\psi(s_n, t_n) \rightarrow \psi(s, t)\) as \(\psi\) is continuous. As \( k \) is continuous,
\[
\lim_{n \to \infty} k \circ \psi(s_n, t_n) = \lim_{n \to \infty} k[\psi(s_n, t_n)]
= k[\lim_{n \to \infty} \psi(s_n, t_n)] = k[\psi(s, t)] = k \circ \psi(s, t),
\]
which proves that \( k \circ \psi \) is continuous. Hence \( f \circ \alpha \) and \( g \circ \beta \) are continuous.

Now define the function \( H : \mathbb{R}^2 \rightarrow \mathbb{R} \) by
\[
H(s, t) = f \circ \alpha(s, t)g \circ \beta(s, t) = f(s - t)g(t).
\]
As \( H \) is the product of two measurable functions, it is measurable (Handbook, page 26).

We use the translational invariance of Lebesgue measure, as given in the question:
\[
|f(s)| = |f(s - t)| \text{ is integrable with respect to } s,
\]
with
\[
\int |f(s - t)| \, ds = \int |f(s)| \, ds = \|f\|.
\]

So
\[
\int |H(s, t)| \, ds
\]
exists, as it equals
\[
|g(t)| \int |f(s - t)| \, ds = |g(t)| \|f\|.
\]

Then
\[
\int \left( \int |H(s, t)| \, ds \right) \, dt = \|f\| \int |g(t)| \, dt = \|f\| \|g\|.
\]

By Tonelli’s Theorem, W Theorem 6.1.2 (Handbook, page 25), \( H \) is integrable on \( \mathbb{R}^2 \), so the integral
\[
\int_{\mathbb{R}^2} f(s - t)g(t) \, d(s, t)
\]
is well defined and equals
\[
\int \left( \int f(s - t)g(t) \, ds \right) \, dt = \int \left( \int f(s - t)g(t) \, dt \right) \, ds.
\]
Hence
\[ h(s) = \int H(s, t) \, dt \]
extists for almost all \( s \) and is integrable.

Finally,
\[
\|h\| = \int |h(s)| \, ds = \int \left( \int f(s - t)g(t) \, dt \right) \, ds \leq \int \left( \int |f(s - t)g(t)| \, dt \right) \, ds = \int \left( \int |f(s - t)g(t)| \, ds \right) \, dt = \|f\| \|g\|.
\]

**Question 15**

(a) By condition (N4) \((\text{Handbook, page 27})\)
\[
\|f + g\| \leq \|f\| + \|g\|.
\]
By a simple finite induction, for a sequence of functions in \( L^2(\mathbb{R}) \),
\[
\left\| \sum_{k=1}^{n} f_k \right\| \leq \sum_{k=1}^{n} \|f_k\|.
\]
Recall that if \( f \in L^2(\mathbb{R}) \) then \( |f| \in L^2(\mathbb{R}) \).

It will prove convenient to write
\[
h_n(x) = \left( \sum_{k=1}^{n} |f_k(x)| \right)^2.
\]
As \( L^2(\mathbb{R}) \) is a vector space, \( h_n \) is integrable, and
\[
\int h_n = \left\| \sum_{k=1}^{n} |f_k| \right\|^2.
\]
Applying the inequality (N4) \((\text{Handbook, page 27})\)
\[
\left\| \sum_{k=1}^{n} |f_k| \right\|^2 \leq \left( \sum_{k=1}^{n} \|f_k\| \right)^2.
\]
Because the series \( \sum \|f_k\| \) is positive, increasing and bounded above by its limit, \( A \),
\[
\left( \sum_{k=1}^{n} \|f_k\| \right)^2 \leq A^2.
\]
This gives the required inequality:
\[
\int h_n \leq A^2.
\]

(b) Consider the sequence of functions \( \{h_n\} \). It is evidently increasing:
\[
h_{n+1} = h_n + |f_{n+1}|^2 + 2|f_{n+1}|h_n \geq h_n.
\]
In part (a) we have shown that the sequence of integrals \( \{\int h_n\} \) was bounded above by \( A^2 \). Thus, the sequence of functions \( \{h_n\} \) is integrable, increasing and the sequence of its integrals is bounded above. The Monotone Convergence Theorem tells us that \( \{h_n\} \) converges a.e.; hence \( \sum_{k=1}^{\infty} f_k \) converges absolutely a.e. Let us write
\[
h(x) = \begin{cases} 
\lim_{n \to \infty} h_n(x) & \text{if the sequence converges}, \\
0 & \text{otherwise}.
\end{cases}
\]
Then by the Monotone Convergence Theorem, \( h \) is integrable and
\[
\int h = \lim_{n \to \infty} \int h_n \leq \lim_{n \to \infty} A^2 = A^2.
\]
We observe that the conditions of the Dominated Convergence Theorem are satisfied for $f^2$: it is the limit a.e. of the sequence of functions $\{g_n\}$, where

$$g_n = \left( \sum_{k=1}^{n} f_k \right)^2.$$ 

We are assured of this since the convergence of $\{h_n\}$ implies the convergence of $\{g_n\}$. (If a series $\sum |a_n|$ converges, then $\sum a_n$ converges.)

The functions $g_n$ are bounded by an integrable function:

$$g_n \leq h,$$

a.e.

Then $f^2$ is integrable, and

$$\|f\|^2 = \int f^2 = \lim_{n \to \infty} \int g_n \leq \int h \leq A^2.$$ 

Taking (positive) square roots,

$$\|f\| = \lim_{n \to \infty} \left\| \sum_{k=1}^{n} f_k \right\| \leq A.$$

This completes the demonstration.